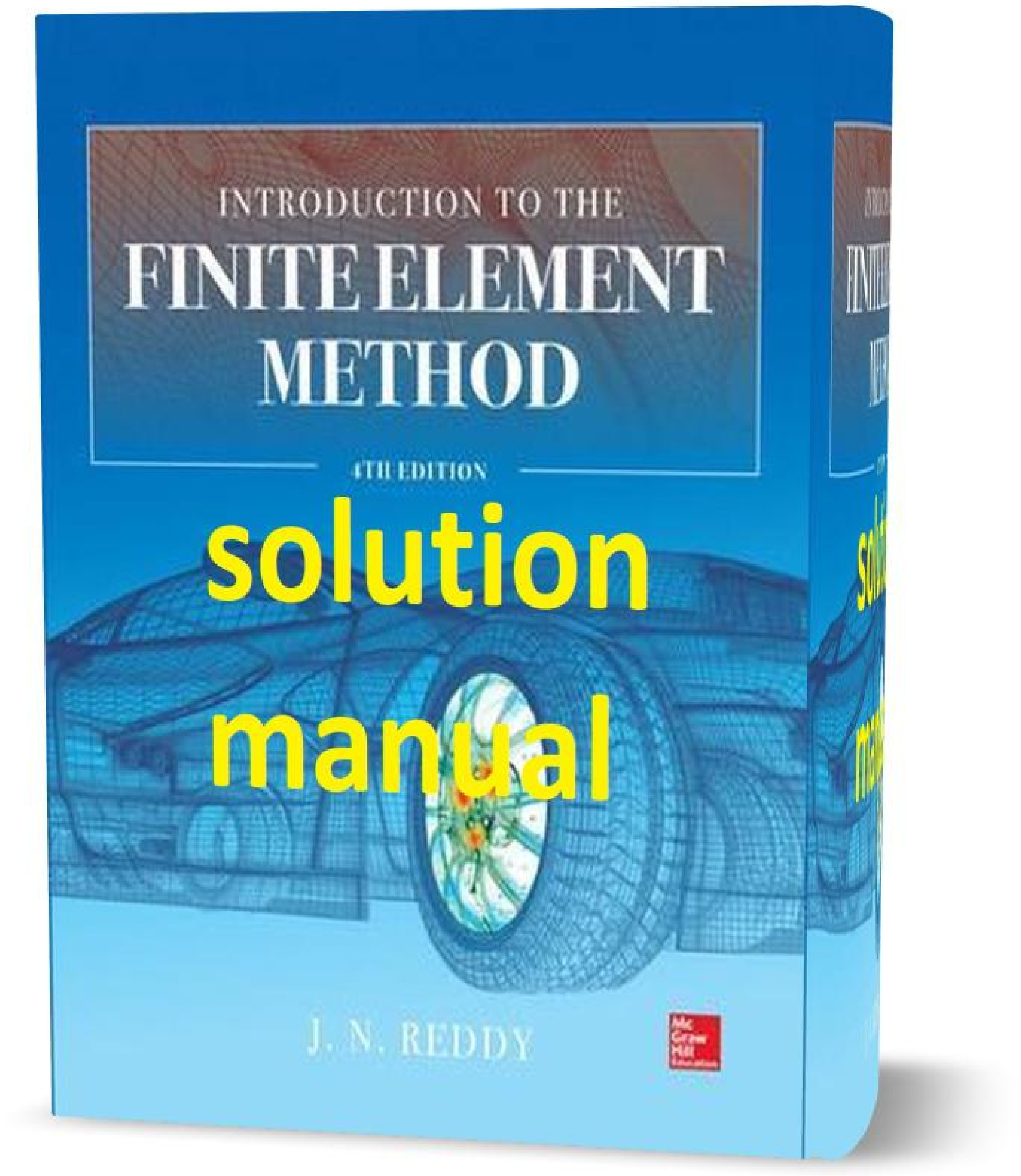
INTRODUCTION TO THE
FINITE ELEMENT
METHOD
4TH EDITION
solution
manual
J. N. REDDY
Mc
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Education

Solutions' Manual

Introduction to the Finite Element Method

Theory and Applications

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Contents

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Preface

This solution manual is prepared to aid the instructor in discussing the solutions to assigned problems in Chapters 1 through 13 from the book, *Introduction to the Finite Element Method,* Fourth Edition, McGraw–Hill, New York, 2019.

The instructor should make an effort to review the problems before assigning them. This allows the instructor to make comments and suggestions on the approach to be taken and nature of the answers expected. The instructor may wish to generate additional problems from those given in this book, especially when taught time and again from the same book. Suggestions for new problems are also included at pertinent places in this manual.

The computer problems **FEM1D** and **FEM2D** can be readily modified to solve new types of field problems. The programs can be easily extended to finite element models formulated in an advanced course and/or in research. The Fortran sources of **FEM1D** and **FEM2D** are available from the author's website http://mechanics.tamu.edu.

The author appreciates receiving comments on the book and a list of errors found in the book and this solutions manual.

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1 General Introduction

Mathematics is the language with which God has written the universe.

— Galileo Galilei

1.1 Newton's second law can be expressed as

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the net force acting on the body, m mass of the body, and $\mathbf{a}$ is the acceleration of the body in the direction of the net force. Determine the mathematical model, that is, the governing equation of a free-falling body. Consider only the forces due to gravity and the air resistance. Assume that the air resistance is linearly proportional to the velocity of the falling body.

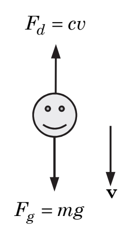

Fig. P1.1

Solution: From the free-body-diagram it follows that

$$m\frac{dv}{dt} = F_g - F_d, \quad F_g = mg, \quad F_d = cv$$

where v is the downward velocity (m/s) of the body, F_g is the downward force (N or kg m/s^2) due to gravity, F_d is the upward drag force, m is the mass (kg) of the body, g the acceleration (m/s^2) due to gravity, and c is the proportionality constant (drag coefficient, kg/s). The equation of motion is

$$\frac{dv}{dt} + \alpha v = g, \quad \alpha = \frac{c}{m}$$

1.2 A cylindrical storage tank of diameter D contains a liquid at depth (or head) $h(x,t)$. Liquid is supplied to the tank at a rate of q_i (m^3/day) and drained at a rate of q_0 (m^3/day). Use the principle of conservation of mass to arrive at the governing equation of the flow problem.

Solution: The conservation of mass requires

$$\text{time rate of change in mass} = \text{mass inflow - mass outflow}$$

The above equation for the problem at hand becomes

$$\frac{d}{dt}(\rho A h) = \rho q_i - \rho q_0 \quad \text{or} \quad \frac{d(Ah)}{dt} = q_i - q_0$$

where A is the area of cross section of the tank ($A = \pi D^2/4$) and ρ is the mass density of the liquid.

1.3 Consider the simple pendulum of **Example 1.3.1**. Write a computer program to numerically solve the *linear* equation (1.2.4) using Euler's (or forward difference) finite difference scheme. Tabulate the numerical results for two different time steps $\Delta t = 0.05$ and $\Delta t = 0.025$ along with the exact linear solution.

Solution: One needs to program the finite difference equations in Eq. (1.3.6). Table 1.3 contains representative numerical results.

Table 1.3 Comparison of various approximate solutions of the equation $(d^2\theta/dt^2) + \lambda^2\theta = 0$ with its exact linear solution.

t	Exact	Approx. solution, θ		Exact	Approx. solution, v	
	θ	$\Delta t = 0.05$	$\Delta t = 0.025$	v	$\Delta t = 0.05$	$\Delta t = 0.025$
0.00	0.78540	0.78540	0.78540	−0.00000	−0.00000	−0.00000
0.05	0.76965	0.78540	0.77749	−0.62801	−0.63225	−0.63225
0.10	0.72302	0.75379	0.73806	−1.23083	−1.26449	−1.25177
0.15	0.64739	0.69056	0.66804	−1.78428	−1.87129	−1.83331
0.20	0.54578	0.59700	0.56966	−2.26615	−2.42719	−2.35264
0.25	0.42229	0.47564	0.44629	−2.65711	−2.90777	−2.78754
0.30	0.28185	0.33025	0.30243	−2.94148	−3.29066	−3.11875
0.35	0.13011	0.16572	0.14344	−3.10785	−3.55651	−3.33082
0.40	−0.02685	−0.01211	−0.02454	−3.14955	−3.68991	−3.41278
0.45	−0.18274	−0.19661	−0.19493	−3.06491	−3.68016	−3.35868
0.50	−0.33129	−0.38061	−0.36091	−2.85732	−3.52190	−3.16797
0.60	−0.58310	−0.71748	−0.65276	−2.11119	−2.76735	−2.40181
0.80	−0.78356	−1.07657	−0.92032	0.21536	0.11379	0.21327
1.00	−0.50591	−0.79648	−0.62784	2.41051	3.41351	2.91148

1.4 Consider the simple pendulum of **Example 1.3.1**. Write a computer program to numerically solve the *nonlinear* equation (1.2.3) using Euler's (or forward difference) finite difference scheme. Tabulate the numerical results for two different time steps $\Delta t = 0.05$ and $\Delta t = 0.025$ along with the exact linear solution.

Solution: In order to use the forward finite difference scheme in Eq. (1.3.2), we rewrite the nonlinear equation in (1.2.3) as a pair of first-order equations

$$\frac{d\theta}{dt} = v, \quad \frac{dv}{dt} = -\lambda^2 \sin\theta$$

Applying the scheme of Eq. (1.3.2) to the two equations at hand, we obtain

$$\theta_{i+1} = \theta_i + \Delta t\ v_i; \quad v_{i+1} = v_i - \Delta t\ \lambda^2 \sin\theta_i$$

The above equations can be programmed to solve for (θ_i, v_i). Table 1.4 contains representative numerical results.

1.5 An improvement of Euler's method is provided by Heun's method, which uses the average of the derivatives at the two ends of the interval to estimate the slope. Applied to the equation

$$\frac{du}{dt} = f(t, u)$$

Heun's scheme has the form

$$u_{i+1} = u_i + \frac{\Delta t}{2}\left[f(t_i, u_i) + f(t_{i+1}, u^0_{i+1})\right], \quad u^0_{i+1} = u_i + \Delta t\ f(t_i, u_i)$$

Table 1.4 Comparison of various approximate solutions of the equation $(d^2\theta/dt^2) + \lambda^2 \sin\theta = 0$ with exact linear solution.

	Exact	Approx. solution, θ		Exact	Approx. solution, v	
t	θ	Euler's	Heun's	v	Euler's	Heun's
0.00	0.78540	0.78540	0.78540	−0.00000	−0.00000	−0.00000
0.05	0.76965	0.78540	0.77828	−0.62801	−0.56922	−0.56922
0.10	0.72302	0.75694	0.74276	−1.23083	−1.13844	−1.13027
0.15	0.64739	0.70002	0.67944	−1.78428	−1.69123	−1.66622
0.20	0.54578	0.58980	0.56482	−2.26615	−2.20984	−2.15879
0.25	0.42229	0.50496	0.47627	−2.65711	−2.67459	−2.58816
0.30	0.28185	0.37123	0.34225	−2.94148	−3.06403	−2.93371
0.35	0.13011	0.21803	0.19218	−3.10785	−3.35605	−3.17573
0.40	−0.02685	0.05023	0.03148	−3.14955	−3.53018	−3.29791
0.45	−0.18274	−0.12628	−0.13374	−3.06491	−3.57060	−3.29007
0.50	−0.33129	−0.30481	−0.29690	−2.85732	−3.46921	−3.15014
0.60	−0.58310	−0.63965	−0.59131	−2.11119	−2.85712	−2.50787
0.80	−0.78356	−1.05068	−0.91171	0.21536	−0.50399	−0.28356
1.00	−0.50591	−0.94062	−0.74672	2.41051	2.29398	2.19765

The second equation is known as the *predictor* equation and the first equation is called the *corrector* equation. Apply Heun's method to Eq. (1.3.5) and obtain the numerical solution for $\Delta t = 0.05$.

Solution: Heun's method applied to the pair

$$\frac{d\theta}{dt} = v, \quad \frac{dv}{dt} = -\lambda^2 \sin\theta$$

yields the following discrete equations:

$$\theta^0_{i+1} = \theta_i + \Delta t\ v_i$$

$$v_{i+1} = v_i - \lambda^2 \frac{\Delta t}{2}\left(\sin\theta_i + \sin\theta^0_{i+1}\right)$$

$$\theta_{i+1} = \theta_i + \frac{\Delta t}{2}\left(v_i + v_{i+1}\right)$$

The numereical results obtained with the Heun's method and Euler's method are presented in Table 1.5.

Table 1.5 Numerical solutions of the nonlinear equation $d^2\theta/dt^2 + \lambda^2 \sin\theta = 0$ along with the exact solution of the linear equation $d^2\theta/dt^2 + \lambda^2\theta = 0$.

	Exact	Approx. solution, θ		Exact	Approx. solution, v	
t	θ	Euler's	Heun's	v	Euler's	Heun's
0.00	0.785398	0.785398	0.785398	−0.000000	−0.000000	−0.000000
0.05	0.769645	0.785398	0.771168	−0.628013	−0.569221	−0.569221
0.10	0.723017	0.756937	0.728680	−1.230833	−1.138442	−1.121957
0.20	0.545784	0.615453	0.564818	−2.266146	−2.209838	−1.121957
0.40	−0.026852	0.050228	0.015246	−3.149552	−3.530178	−3.073095
0.60	−0.583104	−0.639652	−0.544352	−2.111190	−2.857121	−2.194398
0.80	−0.783562	−1.050679	−0.787095	0.215362	−0.503993	−0.114453
1.00	−0.505912	−0.940622	−0.587339	2.410506	2.293983	2.023807

1.6 Show that the backward difference approximation of the boundary condition in Eq. (1.2.20) yields

$$\theta_{N+1} = \left(1 + \frac{\beta\Delta x}{k}\right)^{-1} \theta_N$$

and that it is the same as that in Eq. (1.3.9) when $\frac{\beta\Delta x}{k} < 1$.

Solution: Expanding in Taylor's series and keeping the first two terms, we obtain the required result:

$$\theta_{N+1} = \left(1 + \frac{\beta\Delta x}{k}\right)^{-1} \theta_N \approx \left(1 - \frac{\beta\Delta x}{k}\right) \theta_N$$

1.7 Write a computer program to solve the rod problem of **Example 1.3.2** using 8 intervals (i.e., $\Delta x = 0.00625$) and determine the solution at mesh points $x = 0.00625$, 0.0125, 0.01875, $\cdots$, 0.05 m.

Solution: For a subdivision of 8 subintervals, we have $\Delta x = 0.00625$, $D = 2 + (20 \times 0.00625)^2 = 2.015625$, and $D - 1 + (\beta\Delta x/k) = 1.028125$. Hence, the finite difference equations are

$$\begin{aligned}
2.015625\,\theta_1 \quad -\theta_2 \quad\quad &= 300\\
-\theta_1 \quad +2.015625\,\theta_2 \quad -\theta_3 \quad\quad &= 0\\
-\theta_2 \quad +2.015625\,\theta_3 \quad -\theta_4 \quad\quad &= 0\\
\vdots \quad\quad\quad \vdots \quad\quad &\\
-\theta_6 \quad +2.015625\,\theta_7 \quad -\theta_8 &= 0\\
-\theta_7 \quad +1.028125\,\theta_8 &= 0
\end{aligned}$$

The finite difference solutions is

$$\{\theta\} = \{271.46,\ 247.16,\ 226.72,\ 209.82,\ 196.21,\ 185.66,\ 178.00, 173.13\}^{\mathrm{T}}$$

The analytical solution at the same points is

$$\{\theta\} = \{272.25,\ 248.75,\ 229.15,\ 213.13,\ 200.44,\ 190.90,\ 184.33,\ 180.66\}^{\mathrm{T}}$$

1.8 Repeat **Problem 1.7** for 16 subdivisions and compare the finite difference solution with the analytical solution.

Solution: For a subdivision of 16 subintervals, we have $\Delta x = 0.003125$, $D = 2 + (20 \times 0.003125)^2 = 2.00390625$, and $D - 1 + (\beta \Delta x/k) = 1.01015625$. Hence, the finite difference equations are

$$\begin{array}{rrrrrrl}
2.00391\,\theta_1 & -\theta_2 & & & & & = 300 \\
-\theta_1 & 2.00391\,\theta_2 & -\theta_3 & & & & = \ \ 0 \\
 & -\theta_2 & 2.00391\,\theta_3 & -\theta_4 & & & = \ \ 0 \\
 & & \vdots & & & \vdots & \\
 & & -\theta_{13} & 2.00391\,\theta_{14} & & -\theta_{15} & = \ \ 0 \\
 & & & -\theta_{15} & & 1.01016\,\theta_{16} & = \ \ 0
\end{array}$$

The solution of these equations is

$$\{\theta\} = \{285.36,\ 271.84,\ 259.37,\ 247.92,\ 237.44,\ 227.89,\ 219.22, 211.42,\ \cdots, 176.78\}^{\mathrm{T}}$$

The analytical solution at the same points is

$$\{\theta\} = \{285.56,\ 272.25,\ 259.99,\ 248.75,\ 238.48,\ 229.15,\ 220.71, 213.13,\ \cdots, 180.66\}^{\mathrm{T}}$$
