

Chapter 2

2.1

Sketch

2.2

Sketch

2.3

Sketch

2.4

From Problem 2.2, phase = $\frac{2\pi x}{\lambda} - \omega t$
= constant

Then

$$\frac{2\pi}{\lambda} \cdot \frac{dx}{dt} - \omega = 0, \Rightarrow \frac{dx}{dt} = v_p = +\omega \left(\frac{\lambda}{2\pi} \right)$$

From Problem 2.3, phase = $\frac{2\pi x}{\lambda} + \omega t$
= constant

Then

$$\frac{2\pi}{\lambda} \cdot \frac{dx}{dt} + \omega = 0, \Rightarrow \frac{dx}{dt} = v_p = -\omega \left(\frac{\lambda}{2\pi} \right)$$

2.5

$$E = h\nu = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E}$$

Gold: $E = 4.90 \text{ eV} = (4.90)(1.6 \times 10^{-19}) \text{ J}$

So,

$$\lambda = \frac{(6.625 \times 10^{-34})(3 \times 10^{10})}{(4.90)(1.6 \times 10^{-19})} = 2.54 \times 10^{-5} \text{ cm}$$

or

$$\lambda = 0.254 \mu\text{m}$$

Cesium: $E = 1.90 \text{ eV} = (1.90)(1.6 \times 10^{-19}) \text{ J}$

So,

$$\lambda = \frac{(6.625 \times 10^{-34})(3 \times 10^{10})}{(1.90)(1.6 \times 10^{-19})} = 6.54 \times 10^{-5} \text{ cm}$$

or

$$\lambda = 0.654 \mu\text{m}$$

2.6

$$(a) \quad p = \frac{h}{\lambda} = \frac{6.625 \times 10^{-34}}{550 \times 10^{-9}}$$

$$= 1.205 \times 10^{-27} \text{ kg-m/s}$$

$$v = \frac{p}{m} = \frac{1.2045 \times 10^{-27}}{9.11 \times 10^{-31}} = 1.32 \times 10^3 \text{ m/s}$$

$$\text{or } v = 1.32 \times 10^5 \text{ cm/s}$$

$$(b) \quad p = \frac{h}{\lambda} = \frac{6.625 \times 10^{-34}}{440 \times 10^{-9}}$$

$$= 1.506 \times 10^{-27} \text{ kg-m/s}$$

$$v = \frac{p}{m} = \frac{1.5057 \times 10^{-27}}{9.11 \times 10^{-31}} = 1.65 \times 10^3 \text{ m/s}$$

$$\text{or } v = 1.65 \times 10^5 \text{ cm/s}$$

(c) Yes

2.7

(a) (i)

$$p = \sqrt{2mE} = \sqrt{2(9.11 \times 10^{-31})(1.2)(1.6 \times 10^{-19})}$$

$$= 5.915 \times 10^{-25} \text{ kg-m/s}$$

$$\lambda = \frac{h}{p} = \frac{6.625 \times 10^{-34}}{5.915 \times 10^{-25}} = 1.12 \times 10^{-9} \text{ m}$$

$$\text{or } \lambda = 11.2 \text{ \AA}$$

$$(ii) \quad p = \sqrt{2(9.11 \times 10^{-31})(12)(1.6 \times 10^{-19})}$$

$$= 1.87 \times 10^{-24} \text{ kg-m/s}$$

$$\lambda = \frac{6.625 \times 10^{-34}}{1.8704 \times 10^{-24}} = 3.54 \times 10^{-10} \text{ m}$$

$$\text{or } \lambda = 3.54 \text{ \AA}$$

$$(iii) \quad p = \sqrt{2(9.11 \times 10^{-31})(120)(1.6 \times 10^{-19})}$$

$$= 5.915 \times 10^{-24} \text{ kg-m/s}$$

$$\lambda = \frac{6.625 \times 10^{-34}}{5.915 \times 10^{-24}} = 1.12 \times 10^{-10} \text{ m}$$

$$\text{or } \lambda = 1.12 \text{ \AA}$$

(b)

$$p = \sqrt{2(1.67 \times 10^{-27})(1.2)(1.6 \times 10^{-19})}$$

$$= 2.532 \times 10^{-23} \text{ kg-m/s}$$

$$\lambda = \frac{6.625 \times 10^{-34}}{2.532 \times 10^{-23}} = 2.62 \times 10^{-11} \text{ m}$$

or $\lambda = 0.262 \text{ \AA}$

2.8

$$E_{avg} = \frac{3}{2} kT = \left(\frac{3}{2}\right)(0.0259) = 0.03885 \text{ eV}$$

Now

$$p_{avg} = \sqrt{2mE_{avg}}$$

$$= \sqrt{2(9.11 \times 10^{-31})(0.03885)(1.6 \times 10^{-19})}$$

or

$$p_{avg} = 1.064 \times 10^{-25} \text{ kg-m/s}$$

Now

$$\lambda = \frac{h}{p} = \frac{6.625 \times 10^{-34}}{1.064 \times 10^{-25}} = 6.225 \times 10^{-9} \text{ m}$$

or

$$\lambda = 62.25 \text{ \AA}$$

2.9

$$E_p = h\nu_p = \frac{hc}{\lambda_p}$$

Now

$$E_e = \frac{p_e^2}{2m} \text{ and } p_e = \frac{h}{\lambda_e} \Rightarrow E_e = \frac{1}{2m} \left(\frac{h}{\lambda_e} \right)^2$$

Set $E_p = E_e$ and $\lambda_p = 10\lambda_e$

Then

$$\frac{hc}{\lambda_p} = \frac{1}{2m} \left(\frac{h}{\lambda_e} \right)^2 = \frac{1}{2m} \left(\frac{10h}{\lambda_p} \right)^2$$

which yields

$$\lambda_p = \frac{100h}{2mc}$$

$$E_p = E = \frac{hc}{\lambda_p} = \frac{hc}{100h} \cdot 2mc = \frac{2mc^2}{100}$$

$$= \frac{2(9.11 \times 10^{-31})(3 \times 10^8)^2}{100}$$

$$= 1.64 \times 10^{-15} \text{ J} = 10.25 \text{ keV}$$

2.10

(a) $p = \frac{h}{\lambda} = \frac{6.625 \times 10^{-34}}{85 \times 10^{-10}}$

$$= 7.794 \times 10^{-26} \text{ kg-m/s}$$

$$v = \frac{p}{m} = \frac{7.794 \times 10^{-26}}{9.11 \times 10^{-31}} = 8.56 \times 10^4 \text{ m/s}$$

or $v = 8.56 \times 10^6 \text{ cm/s}$

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(9.11 \times 10^{-31})(8.56 \times 10^4)^2$$

$$= 3.33 \times 10^{-21} \text{ J}$$

or $E = \frac{3.334 \times 10^{-21}}{1.6 \times 10^{-19}} = 2.08 \times 10^{-2} \text{ eV}$

(b) $E = \frac{1}{2}(9.11 \times 10^{-31})(8 \times 10^3)^2$

$$= 2.915 \times 10^{-23} \text{ J}$$

or $E = \frac{2.915 \times 10^{-23}}{1.6 \times 10^{-19}} = 1.82 \times 10^{-4} \text{ eV}$

$$p = mv = (9.11 \times 10^{-31})(8 \times 10^3)$$

$$= 7.288 \times 10^{-27} \text{ kg-m/s}$$

$$\lambda = \frac{h}{p} = \frac{6.625 \times 10^{-35}}{7.288 \times 10^{-27}} = 9.09 \times 10^{-8} \text{ m}$$

or $\lambda = 909 \text{ \AA}$

2.11

(a) $E = h\nu = \frac{hc}{\lambda} = \frac{(6.625 \times 10^{-34})(3 \times 10^8)}{1 \times 10^{-10}}$

$$= 1.99 \times 10^{-15} \text{ J}$$

Now

$$E = e \cdot V \Rightarrow V = \frac{E}{e} = \frac{1.99 \times 10^{-15}}{1.6 \times 10^{-19}}$$

$$V = 1.24 \times 10^4 \text{ V} = 12.4 \text{ kV}$$

(b) $p = \sqrt{2mE} = \sqrt{2(9.11 \times 10^{-31})(1.99 \times 10^{-15})}$

$$= 6.02 \times 10^{-23} \text{ kg-m/s}$$

Then

$$\lambda = \frac{h}{p} = \frac{6.625 \times 10^{-34}}{6.02 \times 10^{-23}} = 1.10 \times 10^{-11} \text{ m}$$

or

$$\lambda = 0.11 \text{ \AA}$$

2.12

$$\Delta p = \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}}{10^{-6}} = 1.054 \times 10^{-28} \text{ kg-m/s}$$

2.13

(a) (i) $\Delta p \Delta x = \hbar$

$$\Delta p = \frac{1.054 \times 10^{-34}}{12 \times 10^{-10}} = 8.783 \times 10^{-26} \text{ kg-m/s}$$

$$\begin{aligned} \text{(ii) } \Delta E &= \frac{dE}{dp} \cdot \Delta p = \frac{d}{dp} \left(\frac{p^2}{2m} \right) \cdot \Delta p \\ &= \frac{2p}{2m} \cdot \Delta p = \frac{p \Delta p}{m} \end{aligned}$$

$$\begin{aligned} \text{Now } p &= \sqrt{2mE} \\ &= \sqrt{2(9 \times 10^{-31})(16)(1.6 \times 10^{-19})} \\ &= 2.147 \times 10^{-24} \text{ kg-m/s} \\ \text{so } \Delta E &= \frac{(2.1466 \times 10^{-24})(8.783 \times 10^{-26})}{9 \times 10^{-31}} \\ &= 2.095 \times 10^{-19} \text{ J} \end{aligned}$$

$$\text{or } \Delta E = \frac{2.095 \times 10^{-19}}{1.6 \times 10^{-19}} = 1.31 \text{ eV}$$

(b) (i) $\Delta p = 8.783 \times 10^{-26} \text{ kg-m/s}$

$$\begin{aligned} \text{(ii) } p &= \sqrt{2(5 \times 10^{-28})(16)(1.6 \times 10^{-19})} \\ &= 5.06 \times 10^{-23} \text{ kg-m/s} \\ \Delta E &= \frac{(5.06 \times 10^{-23})(8.783 \times 10^{-26})}{5 \times 10^{-28}} \\ &= 8.888 \times 10^{-21} \text{ J} \\ \text{or } \Delta E &= \frac{8.888 \times 10^{-21}}{1.6 \times 10^{-19}} = 5.55 \times 10^{-2} \text{ eV} \end{aligned}$$

2.14

$$\Delta p = \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}}{10^{-2}} = 1.054 \times 10^{-32} \text{ kg-m/s}$$

$$p = m v \Rightarrow \Delta v = \frac{\Delta p}{m} = \frac{1.054 \times 10^{-32}}{1500}$$

$$\Delta v = 7 \times 10^{-36} \text{ m/s}$$

2.15

(a) $\Delta E \Delta t = \hbar$

$$\Delta t = \frac{1.054 \times 10^{-34}}{(0.8)(1.6 \times 10^{-19})} = 8.23 \times 10^{-16} \text{ s}$$

$$\begin{aligned} \text{(b) } \Delta p &= \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}}{1.5 \times 10^{-10}} \\ &= 7.03 \times 10^{-25} \text{ kg-m/s} \end{aligned}$$

2.16

(a) If $\Psi_1(x, t)$ and $\Psi_2(x, t)$ are solutions to Schrodinger's wave equation, then

$$\frac{-\hbar^2}{2m} \cdot \frac{\partial^2 \Psi_1(x, t)}{\partial x^2} + V(x) \Psi_1(x, t) = j\hbar \frac{\partial \Psi_1(x, t)}{\partial t}$$

and

$$\frac{-\hbar^2}{2m} \cdot \frac{\partial^2 \Psi_2(x, t)}{\partial x^2} + V(x) \Psi_2(x, t) = j\hbar \frac{\partial \Psi_2(x, t)}{\partial t}$$

Adding the two equations, we obtain

$$\begin{aligned} \frac{-\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} [\Psi_1(x, t) + \Psi_2(x, t)] \\ + V(x) [\Psi_1(x, t) + \Psi_2(x, t)] \\ = j\hbar \frac{\partial}{\partial t} [\Psi_1(x, t) + \Psi_2(x, t)] \end{aligned}$$

which is Schrodinger's wave equation. So $\Psi_1(x, t) + \Psi_2(x, t)$ is also a solution.

(b) If $\Psi_1(x, t) \cdot \Psi_2(x, t)$ were a solution to Schrodinger's wave equation, then we could write

$$\begin{aligned} \frac{-\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} [\Psi_1 \cdot \Psi_2] + V(x) [\Psi_1 \cdot \Psi_2] \\ = j\hbar \frac{\partial}{\partial t} [\Psi_1 \cdot \Psi_2] \end{aligned}$$

which can be written as

$$\begin{aligned} \frac{-\hbar^2}{2m} \left[\Psi_1 \frac{\partial^2 \Psi_2}{\partial x^2} + \Psi_2 \frac{\partial^2 \Psi_1}{\partial x^2} + 2 \frac{\partial \Psi_1}{\partial x} \cdot \frac{\partial \Psi_2}{\partial x} \right] \\ + V(x) [\Psi_1 \cdot \Psi_2] = j\hbar \left[\Psi_1 \frac{\partial \Psi_2}{\partial t} + \Psi_2 \frac{\partial \Psi_1}{\partial t} \right] \end{aligned}$$

Dividing by $\Psi_1 \cdot \Psi_2$, we find

$$\begin{aligned} \frac{-\hbar^2}{2m} \left[\frac{1}{\Psi_2} \cdot \frac{\partial^2 \Psi_2}{\partial x^2} + \frac{1}{\Psi_1} \cdot \frac{\partial^2 \Psi_1}{\partial x^2} + \frac{2}{\Psi_1 \Psi_2} \frac{\partial \Psi_1}{\partial x} \frac{\partial \Psi_2}{\partial x} \right] \\ + V(x) = j\hbar \left[\frac{1}{\Psi_2} \frac{\partial \Psi_2}{\partial t} + \frac{1}{\Psi_1} \frac{\partial \Psi_1}{\partial t} \right] \end{aligned}$$

Since Ψ_1 is a solution, then

$$\frac{-\hbar^2}{2m} \cdot \frac{1}{\Psi_1} \cdot \frac{\partial^2 \Psi_1}{\partial x^2} + V(x) = j\hbar \cdot \frac{1}{\Psi_1} \cdot \frac{\partial \Psi_1}{\partial t}$$

Subtracting these last two equations, we have

$$\begin{aligned} \frac{-\hbar^2}{2m} \left[\frac{1}{\Psi_2} \cdot \frac{\partial^2 \Psi_2}{\partial x^2} + \frac{2}{\Psi_1 \Psi_2} \frac{\partial \Psi_1}{\partial x} \frac{\partial \Psi_2}{\partial x} \right] \\ = j\hbar \cdot \frac{1}{\Psi_2} \cdot \frac{\partial \Psi_2}{\partial t} \end{aligned}$$

Since Ψ_2 is also a solution, we have

$$\frac{-\hbar^2}{2m} \cdot \frac{1}{\Psi_2} \cdot \frac{\partial^2 \Psi_2}{\partial x^2} + V(x) = j\hbar \cdot \frac{1}{\Psi_2} \cdot \frac{\partial \Psi_2}{\partial t}$$

Subtracting these last two equations, we obtain

$$\frac{-\hbar^2}{2m} \cdot \frac{2}{\Psi_1 \Psi_2} \cdot \frac{\partial \Psi_1}{\partial x} \cdot \frac{\partial \Psi_2}{\partial x} - V(x) = 0$$

This equation is not necessarily valid, which means that $\Psi_1 \Psi_2$ is, in general, not a solution to Schrodinger's wave equation.

2.17

$$\int_{-1}^{+3} A^2 \cos^2\left(\frac{\pi x}{2}\right) dx = 1$$

$$A^2 \left[\frac{x}{2} + \frac{\sin(\pi x)}{2\pi} \right]_{-1}^{+3} = 1$$

$$A^2 \left[\frac{3}{2} - \left(\frac{-1}{2} \right) \right] = 1$$

$$\text{so } A^2 = \frac{1}{2}$$

$$\text{or } |A| = \frac{1}{\sqrt{2}}$$

2.18

$$\int_{-1/2}^{+1/2} A^2 \cos^2(n\pi x) dx = 1$$

$$A^2 \left[\frac{x}{2} + \frac{\sin(2n\pi x)}{4n\pi} \right]_{-1/2}^{+1/2} = 1$$

$$A^2 \left[\frac{1}{4} - \left(-\frac{1}{4} \right) \right] = 1 = A^2 \left(\frac{1}{2} \right)$$

$$\text{or } |A| = \sqrt{2}$$

2.19

Note that $\int_0^\infty \Psi \cdot \Psi^* dx = 1$

Function has been normalized.

(a) Now

$$\begin{aligned} P &= \int_0^{a_o/4} \left[\sqrt{\frac{2}{a_o}} \exp\left(\frac{-x}{a_o}\right) \right]^2 dx \\ &= \frac{2}{a_o} \int_0^{a_o/4} \exp\left(\frac{-2x}{a_o}\right) dx \\ &= \frac{2}{a_o} \left(\frac{-a_o}{2} \right) \exp\left(\frac{-2x}{a_o}\right) \Big|_0^{a_o/4} \end{aligned}$$

or

$$P = (-1) \left[\exp\left(\frac{-2a_o}{4a_o}\right) - 1 \right] = 1 - \exp\left(\frac{-1}{2}\right)$$

which yields

$$P = 0.393$$

(b)

$$\begin{aligned} P &= \int_{a_o/4}^{a_o/2} \left[\sqrt{\frac{2}{a_o}} \exp\left(\frac{-x}{a_o}\right) \right]^2 dx \\ &= \frac{2}{a_o} \int_{a_o/4}^{a_o/2} \exp\left(\frac{-2x}{a_o}\right) dx \\ &= \frac{2}{a_o} \left(\frac{-a_o}{2} \right) \exp\left(\frac{-2x}{a_o}\right) \Big|_{a_o/4}^{a_o/2} \end{aligned}$$

or

$$P = (-1) \left[\exp(-1) - \exp\left(\frac{-1}{2}\right) \right]$$

which yields

$$P = 0.239$$

(c)

$$\begin{aligned} P &= \int_0^{a_o} \left[\sqrt{\frac{2}{a_o}} \exp\left(\frac{-x}{a_o}\right) \right]^2 dx \\ &= \frac{2}{a_o} \int_0^{a_o} \exp\left(\frac{-2x}{a_o}\right) dx \\ &= \frac{2}{a_o} \left(\frac{-a_o}{2} \right) \exp\left(\frac{-2x}{a_o}\right) \Big|_0^{a_o} \\ &= (-1) [\exp(-2) - 1] \end{aligned}$$

which yields

$$P = 0.865$$

2.20

$$P = \int |\psi(x)|^2 dx$$

$$\begin{aligned} \text{(a)} \quad & \int_0^{a/4} \left(\frac{2}{a}\right) \cos^2\left(\frac{\pi x}{2}\right) dx \\ &= \left(\frac{2}{a}\right) \left[\frac{x}{2} + \frac{\sin\left(\frac{2\pi x}{a}\right)}{4\left(\frac{\pi}{a}\right)} \right]_0^{a/4} \\ &= \left(\frac{2}{a}\right) \left[\left(\frac{a}{4}\right) \frac{1}{2} + \frac{\sin\left(\frac{\pi}{2}\right)}{\left(\frac{4\pi}{a}\right)} \right] \\ &= \left(\frac{2}{a}\right) \left[\frac{a}{8} + \frac{(1)(a)}{4\pi} \right] \end{aligned}$$

or $P = 0.409$

$$\begin{aligned} \text{(b)} \quad & P = \int_{a/4}^{a/2} \left(\frac{2}{a}\right) \cos^2\left(\frac{\pi x}{a}\right) dx \\ &= \left(\frac{2}{a}\right) \left[\frac{x}{2} + \frac{\sin\left(\frac{2\pi x}{a}\right)}{4\left(\frac{\pi}{a}\right)} \right]_{a/4}^{a/2} \\ &= \left(\frac{2}{a}\right) \left[\frac{a}{4} + \frac{\sin(\pi)}{\left(\frac{4\pi}{a}\right)} - \frac{a}{8} - \frac{\sin\left(\frac{\pi}{2}\right)}{\left(\frac{4\pi}{a}\right)} \right] \\ &= 2 \left[\frac{1}{4} + 0 - \frac{1}{8} - \frac{1}{4\pi} \right] \end{aligned}$$

or $P = 0.0908$

$$\begin{aligned} \text{(c)} \quad & P = \int_{-a/2}^{+a/2} \left(\frac{2}{a}\right) \cos^2\left(\frac{\pi x}{a}\right) dx \\ &= \left(\frac{2}{a}\right) \left[\frac{x}{2} + \frac{\sin\left(\frac{2\pi x}{a}\right)}{\left(\frac{4\pi}{a}\right)} \right]_{-a/2}^{+a/2} \\ &= \left(\frac{2}{a}\right) \left[\frac{a}{4} + \frac{\sin(\pi)}{\left(\frac{4\pi}{a}\right)} - \left(\frac{-a}{4}\right) - \frac{\sin(-\pi)}{\left(\frac{4\pi}{a}\right)} \right] \end{aligned}$$

or $P = 1$

2.21

$$\begin{aligned} \text{(a)} \quad & P = \int_0^{a/4} \left(\frac{2}{a}\right) \sin^2\left(\frac{2\pi x}{a}\right) dx \\ &= \left(\frac{2}{a}\right) \left[\frac{x}{2} - \frac{\sin\left(\frac{4\pi x}{a}\right)}{4\left(\frac{2\pi}{a}\right)} \right]_0^{a/4} \\ &= \left(\frac{2}{a}\right) \left[\frac{a}{8} - \frac{\sin(\pi)}{\left(\frac{8\pi}{a}\right)} \right] \end{aligned}$$

or $P = 0.25$

$$\begin{aligned} \text{(b)} \quad & P = \int_{a/4}^{a/2} \left(\frac{2}{a}\right) \sin^2\left(\frac{2\pi x}{a}\right) dx \\ &= \left(\frac{2}{a}\right) \left[\frac{x}{2} - \frac{\sin\left(\frac{4\pi x}{a}\right)}{4\left(\frac{2\pi}{a}\right)} \right]_{a/4}^{a/2} \\ &= \left(\frac{2}{a}\right) \left[\frac{a}{4} - \frac{\sin(2\pi)}{\left(\frac{8\pi}{a}\right)} - \left(\frac{a}{8}\right) + \frac{\sin(\pi)}{\left(\frac{8\pi}{a}\right)} \right] \end{aligned}$$

or $P = 0.25$

$$\begin{aligned} \text{(c)} \quad & P = \int_{-a/2}^{+a/2} \left(\frac{2}{a}\right) \sin^2\left(\frac{2\pi x}{a}\right) dx \\ &= \left(\frac{2}{a}\right) \left[\frac{x}{2} - \frac{\sin\left(\frac{4\pi x}{a}\right)}{4\left(\frac{2\pi}{a}\right)} \right]_{-a/2}^{+a/2} \\ &= \left(\frac{2}{a}\right) \left[\frac{a}{4} - \frac{\sin(2\pi)}{\left(\frac{8\pi}{a}\right)} - \left(\frac{-a}{4}\right) + \frac{\sin(-2\pi)}{\left(\frac{8\pi}{a}\right)} \right] \end{aligned}$$

or $P = 1$

2.22

$$\text{(a) (i)} \quad v_p = \frac{\omega}{k} = \frac{8 \times 10^{12}}{8 \times 10^8} = 10^4 \text{ m/s}$$

or $v_p = 10^6 \text{ cm/s}$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{8 \times 10^8} = 7.854 \times 10^{-9} \text{ m}$$

or $\lambda = 78.54 \text{ \AA}$

(ii) $p = mv = (9.11 \times 10^{-31})(10^4)$
 $= 9.11 \times 10^{-27} \text{ kg-m/s}$
 $E = \frac{1}{2}mv^2 = \frac{1}{2}(9.11 \times 10^{-31})(10^4)^2$
 $= 4.555 \times 10^{-23} \text{ J}$
or $E = \frac{4.555 \times 10^{-23}}{1.6 \times 10^{-19}} = 2.85 \times 10^{-4} \text{ eV}$

(b) (i) $v_p = \frac{\omega}{k} = \frac{1.5 \times 10^{13}}{-1.5 \times 10^9} = -10^4 \text{ m/s}$
or $v_p = -10^6 \text{ cm/s}$
 $\lambda = \frac{2\pi}{|k|} = \frac{2\pi}{1.5 \times 10^9} = 4.19 \times 10^{-9} \text{ m}$

or $\lambda = 41.9 \text{ \AA}$

(ii) $p = -9.11 \times 10^{-27} \text{ kg-m/s}$
 $E = 2.85 \times 10^{-4} \text{ eV}$

2.23

(a) $\Psi(x, t) = Ae^{-j(kx + \omega t)}$

(b) $E = (0.025)(1.6 \times 10^{-19}) = \frac{1}{2}mv^2$
 $= \frac{1}{2}(9.11 \times 10^{-31})v^2$
so $|v| = 9.37 \times 10^4 \text{ m/s} = 9.37 \times 10^6 \text{ cm/s}$
For electron traveling in $-x$ direction,
 $v = -9.37 \times 10^6 \text{ cm/s}$
 $p = mv = (9.11 \times 10^{-31})(-9.37 \times 10^4)$
 $= -8.537 \times 10^{-26} \text{ kg-m/s}$
 $\lambda = \frac{h}{|p|} = \frac{6.625 \times 10^{-34}}{8.537 \times 10^{-26}} = 7.76 \times 10^{-9} \text{ m}$
 $k = \frac{2\pi}{\lambda} = \frac{2\pi}{7.76 \times 10^{-9}} = 8.097 \times 10^8 \text{ m}^{-1}$
 $\omega = k \cdot |v| = (8.097 \times 10^8)(9.37 \times 10^4)$
or $\omega = 7.586 \times 10^{13} \text{ rad/s}$

2.24

(a) $p = mv = (9.11 \times 10^{-31})(5 \times 10^4)$
 $= 4.555 \times 10^{-26} \text{ kg-m/s}$
 $\lambda = \frac{h}{p} = \frac{6.625 \times 10^{-34}}{4.555 \times 10^{-26}} = 1.454 \times 10^{-8} \text{ m}$

$k = \frac{2\pi}{\lambda} = \frac{2\pi}{1.454 \times 10^{-8}} = 4.32 \times 10^8 \text{ m}^{-1}$
 $\omega = kv = (4.32 \times 10^8)(5 \times 10^4)$
 $= 2.16 \times 10^{13} \text{ rad/s}$

(b) $p = (9.11 \times 10^{-31})(10^6)$
 $= 9.11 \times 10^{-25} \text{ kg-m/s}$
 $\lambda = \frac{6.625 \times 10^{-34}}{9.11 \times 10^{-25}} = 7.27 \times 10^{-10} \text{ m}$
 $k = \frac{2\pi}{\lambda} = \frac{2\pi}{7.27 \times 10^{-10}} = 8.64 \times 10^9 \text{ m}^{-1}$
 $\omega = (8.64 \times 10^9)(10^6) = 8.64 \times 10^{15} \text{ rad/s}$

2.25

$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2} = \frac{n^2 (1.054 \times 10^{-34})^2 \pi^2}{2(9.11 \times 10^{-31})(75 \times 10^{-10})^2}$
 $E_n = n^2 (1.0698 \times 10^{-21}) \text{ J}$
or
 $E_n = \frac{n^2 (1.0698 \times 10^{-21})}{1.6 \times 10^{-19}}$
or $E_n = n^2 (6.686 \times 10^{-3}) \text{ eV}$
Then
 $E_1 = 6.69 \times 10^{-3} \text{ eV}$
 $E_2 = 2.67 \times 10^{-2} \text{ eV}$
 $E_3 = 6.02 \times 10^{-2} \text{ eV}$

2.26

(a) $E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2} = \frac{n^2 (1.054 \times 10^{-34})^2 \pi^2}{2(9.11 \times 10^{-31})(10 \times 10^{-10})^2}$
 $= n^2 (6.018 \times 10^{-20}) \text{ J}$
or $E_n = \frac{n^2 (6.018 \times 10^{-20})}{1.6 \times 10^{-19}} = n^2 (0.3761) \text{ eV}$
Then
 $E_1 = 0.376 \text{ eV}$
 $E_2 = 1.504 \text{ eV}$
 $E_3 = 3.385 \text{ eV}$

(b) $\lambda = \frac{hc}{\Delta E}$
 $\Delta E = (3.385 - 1.504)(1.6 \times 10^{-19})$
 $= 3.01 \times 10^{-19} \text{ J}$

$$\lambda = \frac{(6.625 \times 10^{-34})(3 \times 10^8)}{3.01 \times 10^{-19}}$$

$$= 6.604 \times 10^{-7} \text{ m}$$

or $\lambda = 660.4 \text{ nm}$

2.27

(a) $E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$

$$15 \times 10^{-3} = \frac{n^2 (1.054 \times 10^{-34})^2 \pi^2}{2(15 \times 10^{-3})(1.2 \times 10^{-2})^2}$$

$$15 \times 10^{-3} = n^2 (2.538 \times 10^{-62})$$

or $n = 7.688 \times 10^{29}$

(b) $E_{n+1} \cong 15 \text{ mJ}$

(c) No

2.28

For a neutron and $n = 1$:

$$E_1 = \frac{\hbar^2 \pi^2}{2ma^2} = \frac{(1.054 \times 10^{-34})^2 \pi^2}{2(1.66 \times 10^{-27})(10^{-14})^2}$$

$$= 3.3025 \times 10^{-13} \text{ J}$$

or

$$E_1 = \frac{3.3025 \times 10^{-13}}{1.6 \times 10^{-19}} = 2.06 \times 10^6 \text{ eV}$$

For an electron in the same potential well:

$$E_1 = \frac{(1.054 \times 10^{-34})^2 \pi^2}{2(9.11 \times 10^{-31})(10^{-14})^2}$$

$$= 6.0177 \times 10^{-10} \text{ J}$$

or

$$E_1 = \frac{6.0177 \times 10^{-10}}{1.6 \times 10^{-19}} = 3.76 \times 10^9 \text{ eV}$$

2.29

Schrodinger's time-independent wave equation

$$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{2m}{\hbar^2} (E - V(x)) \psi(x) = 0$$

We know that

$$\psi(x) = 0 \text{ for } x \geq \frac{a}{2} \text{ and } x \leq \frac{-a}{2}$$

We have

$$V(x) = 0 \text{ for } \frac{-a}{2} < x < \frac{+a}{2}$$

so in this region

$$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{2mE}{\hbar^2} \psi(x) = 0$$

The solution is of the form

$$\psi(x) = A \cos kx + B \sin kx$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

Boundary conditions:

$$\psi(x) = 0 \text{ at } x = \frac{-a}{2}, x = \frac{+a}{2}$$

First mode solution:

$$\psi_1(x) = A_1 \cos k_1 x$$

where

$$k_1 = \frac{\pi}{a} \Rightarrow E_1 = \frac{\pi^2 \hbar^2}{2ma^2}$$

Second mode solution:

$$\psi_2(x) = B_2 \sin k_2 x$$

where

$$k_2 = \frac{2\pi}{a} \Rightarrow E_2 = \frac{4\pi^2 \hbar^2}{2ma^2}$$

Third mode solution:

$$\psi_3(x) = A_3 \cos k_3 x$$

where

$$k_3 = \frac{3\pi}{a} \Rightarrow E_3 = \frac{9\pi^2 \hbar^2}{2ma^2}$$

Fourth mode solution:

$$\psi_4(x) = B_4 \sin k_4 x$$

where

$$k_4 = \frac{4\pi}{a} \Rightarrow E_4 = \frac{16\pi^2 \hbar^2}{2ma^2}$$

2.30

The 3-D time-independent wave equation in cartesian coordinates for $V(x, y, z) = 0$ is:

$$\frac{\partial^2 \psi(x, y, z)}{\partial x^2} + \frac{\partial^2 \psi(x, y, z)}{\partial y^2} + \frac{\partial^2 \psi(x, y, z)}{\partial z^2} + \frac{2mE}{\hbar^2} \psi(x, y, z) = 0$$

Use separation of variables, so let

$$\psi(x, y, z) = X(x)Y(y)Z(z)$$

Substituting into the wave equation, we obtain

$$YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} + \frac{2mE}{\hbar^2} XYZ = 0$$

Dividing by XYZ and letting $k^2 = \frac{2mE}{\hbar^2}$, we

find

$$(1) \quad \frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \cdot \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \cdot \frac{\partial^2 Z}{\partial z^2} + k^2 = 0$$

We may set

$$\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = -k_x^2 \Rightarrow \frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

Solution is of the form

$$X(x) = A \sin(k_x x) + B \cos(k_x x)$$

Boundary conditions: $X(0) = 0 \Rightarrow B = 0$

$$\text{and } X(x=a) = 0 \Rightarrow k_x = \frac{n_x \pi}{a}$$

where $n_x = 1, 2, 3, \dots$

Similarly, let

$$\frac{1}{Y} \cdot \frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad \text{and} \quad \frac{1}{Z} \cdot \frac{\partial^2 Z}{\partial z^2} = -k_z^2$$

Applying the boundary conditions, we find

$$k_y = \frac{n_y \pi}{a}, \quad n_y = 1, 2, 3, \dots$$

$$k_z = \frac{n_z \pi}{a}, \quad n_z = 1, 2, 3, \dots$$

From Equation (1) above, we have

$$-k_x^2 - k_y^2 - k_z^2 + k^2 = 0$$

or

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \frac{2mE}{\hbar^2}$$

so that

$$E \rightarrow E_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2ma^2} (n_x^2 + n_y^2 + n_z^2)$$

2.31

$$(a) \quad \frac{\partial^2 \psi(x, y)}{\partial x^2} + \frac{\partial^2 \psi(x, y)}{\partial y^2} + \frac{2mE}{\hbar^2} \cdot \psi(x, y) = 0$$

Solution is of the form:

$$\psi(x, y) = A \sin k_x x \cdot \sin k_y y$$

We find

$$\frac{\partial \psi(x, y)}{\partial x} = A k_x \cos k_x x \cdot \sin k_y y$$

$$\frac{\partial^2 \psi(x, y)}{\partial x^2} = -A k_x^2 \sin k_x x \cdot \sin k_y y$$

$$\frac{\partial \psi(x, y)}{\partial y} = A k_y \sin k_x x \cdot \cos k_y y$$

$$\frac{\partial^2 \psi(x, y)}{\partial y^2} = -A k_y^2 \sin k_x x \cdot \sin k_y y$$

Substituting into the original equation, we find:

$$(1) \quad -k_x^2 - k_y^2 + \frac{2mE}{\hbar^2} = 0$$

From the boundary conditions,

$$A \sin k_x a = 0, \quad \text{where } a = 40 \text{ \AA}$$

$$\text{So } k_x = \frac{n_x \pi}{a}, \quad n_x = 1, 2, 3, \dots$$

$$\text{Also } A \sin k_y b = 0, \quad \text{where } b = 20 \text{ \AA}$$

$$\text{So } k_y = \frac{n_y \pi}{b}, \quad n_y = 1, 2, 3, \dots$$

Substituting into Eq. (1) above

$$E_{n_x n_y} = \frac{\hbar^2}{2m} \left(\frac{n_x^2 \pi^2}{a^2} + \frac{n_y^2 \pi^2}{b^2} \right)$$

(b) Energy is quantized - similar to 1-D result.
There can be more than one quantum state per given energy - different than 1-D result.

2.32

(a) Derivation of energy levels exactly the same as in the text

$$(b) \quad \Delta E = \frac{\hbar^2 \pi^2}{2ma^2} (n_2^2 - n_1^2)$$

$$\text{For } n_2 = 2, n_1 = 1$$

Then

$$\Delta E = \frac{3\hbar^2 \pi^2}{2ma^2}$$

(i) For $a = 4 \text{ \AA}$

$$\Delta E = \frac{3(1.054 \times 10^{-34})^2 \pi^2}{2(1.67 \times 10^{-27})(4 \times 10^{-10})^2}$$

$$= 6.155 \times 10^{-22} \text{ J}$$

$$\text{or } \Delta E = \frac{6.155 \times 10^{-22}}{1.6 \times 10^{-19}} = 3.85 \times 10^{-3} \text{ eV}$$

(ii) For $a = 0.5 \text{ cm}$

$$\Delta E = \frac{3(1.054 \times 10^{-34})^2 \pi^2}{2(1.67 \times 10^{-27})(0.5 \times 10^{-2})^2}$$

$$= 3.939 \times 10^{-36} \text{ J}$$

or

$$\Delta E = \frac{3.939 \times 10^{-36}}{1.6 \times 10^{-19}} = 2.46 \times 10^{-17} \text{ eV}$$

2.33

(a) For region II, $x > 0$

$$\frac{\partial^2 \psi_2(x)}{\partial x^2} + \frac{2m}{\hbar^2} (E - V_o) \psi_2(x) = 0$$

General form of the solution is

$$\psi_2(x) = A_2 \exp(jk_2 x) + B_2 \exp(-jk_2 x)$$

where

$$k_2 = \sqrt{\frac{2m}{\hbar^2} (E - V_o)}$$

Term with B_2 represents incident wave and term with A_2 represents reflected wave.

Region I, $x < 0$

$$\frac{\partial^2 \psi_1(x)}{\partial x^2} + \frac{2mE}{\hbar^2} \psi_1(x) = 0$$

General form of the solution is

$$\psi_1(x) = A_1 \exp(jk_1 x) + B_1 \exp(-jk_1 x)$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

Term involving B_1 represents the transmitted wave and the term involving A_1 represents reflected wave: but if a particle is transmitted into region I, it will not be reflected so that $A_1 = 0$.

Then

$$\psi_1(x) = B_1 \exp(-jk_1 x)$$

$$\psi_2(x) = A_2 \exp(jk_2 x) + B_2 \exp(-jk_2 x)$$

(b)

Boundary conditions:

$$(1) \psi_1(x=0) = \psi_2(x=0)$$

$$(2) \left. \frac{\partial \psi_1}{\partial x} \right|_{x=0} = \left. \frac{\partial \psi_2}{\partial x} \right|_{x=0}$$

Applying the boundary conditions to the solutions, we find

$$B_1 = A_2 + B_2$$

$$k_2 A_2 - k_2 B_2 = -k_1 B_1$$

Combining these two equations, we find

$$A_2 = \left(\frac{k_2 - k_1}{k_2 + k_1} \right) \cdot B_2$$

$$B_1 = \left(\frac{2k_2}{k_2 + k_1} \right) \cdot B_2$$

The reflection coefficient is

$$R = \frac{A_2 A_2^*}{B_2 B_2^*} = \left(\frac{k_2 - k_1}{k_2 + k_1} \right)^2$$

The transmission coefficient is

$$T = 1 - R \Rightarrow T = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

2.34

$$\psi_2(x) = A_2 \exp(-k_2 x)$$

$$P = \frac{|\psi(x)|^2}{A_2 A_2^*} = \exp(-2k_2 x)$$

$$\text{where } k_2 = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$$

$$= \frac{\sqrt{2(9.11 \times 10^{-31})(3.5 - 2.8)(1.6 \times 10^{-19})}}{1.054 \times 10^{-34}}$$

$$k_2 = 4.286 \times 10^9 \text{ m}^{-1}$$

(a) For $x = 5 \text{ \AA} = 5 \times 10^{-10} \text{ m}$

$$P = \exp(-2k_2 x)$$

$$= \exp[-2(4.2859 \times 10^9)(5 \times 10^{-10})]$$

$$= 0.0138$$

(b) For $x = 15 \text{ \AA} = 15 \times 10^{-10} \text{ m}$

$$P = \exp[-2(4.2859 \times 10^9)(15 \times 10^{-10})]$$

$$= 2.61 \times 10^{-6}$$

(c) For $x = 40 \text{ \AA} = 40 \times 10^{-10} \text{ m}$

$$P = \exp[-2(4.2859 \times 10^9)(40 \times 10^{-10})]$$

$$= 1.29 \times 10^{-15}$$

2.35

$$T \cong 16 \left(\frac{E}{V_o} \right) \left(1 - \frac{E}{V_o} \right) \exp(-2k_2 a)$$

$$\text{where } k_2 = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$$

$$= \frac{\sqrt{2(9.11 \times 10^{-31})(1.0 - 0.1)(1.6 \times 10^{-19})}}{1.054 \times 10^{-34}}$$

or $k_2 = 4.860 \times 10^9 \text{ m}^{-1}$

(a) For $a = 4 \times 10^{-10} \text{ m}$

$$T \cong 16 \left(\frac{0.1}{1.0} \right) \left(1 - \frac{0.1}{1.0} \right) \exp \left[-2(4.85976 \times 10^9)(4 \times 10^{-10}) \right] \\ = 0.0295$$

(b) For $a = 12 \times 10^{-10} \text{ m}$

$$T \cong 16 \left(\frac{0.1}{1.0} \right) \left(1 - \frac{0.1}{1.0} \right) \exp \left[-2(4.85976 \times 10^9)(12 \times 10^{-10}) \right] \\ = 1.24 \times 10^{-5}$$

(c) $J = N_i e v$, where N_i is the density of transmitted electrons.

$$E = 0.1 \text{ eV} = 1.6 \times 10^{-20} \text{ J}$$

$$= \frac{1}{2} m v^2 = \frac{1}{2} (9.11 \times 10^{-31}) v^2$$

$$\Rightarrow v = 1.874 \times 10^5 \text{ m/s} = 1.874 \times 10^7 \text{ cm/s}$$

$$1.2 \times 10^{-3} = N_i (1.6 \times 10^{-19}) (1.874 \times 10^7)$$

$$N_i = 4.002 \times 10^8 \text{ electrons/cm}^3$$

Density of incident electrons,

$$N_i = \frac{4.002 \times 10^8}{0.0295} = 1.357 \times 10^{10} \text{ cm}^{-3}$$

2.36

$$T \cong 16 \left(\frac{E}{V_o} \right) \left(1 - \frac{E}{V_o} \right) \exp(-2k_2 a)$$

(a) For $m = (0.067)m_o$

$$k_2 = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$$

$$= \left\{ \frac{2(0.067)(9.11 \times 10^{-31})(0.8 - 0.2)(1.6 \times 10^{-19})}{(1.054 \times 10^{-34})^2} \right\}^{1/2}$$

or

$$k_2 = 1.027 \times 10^9 \text{ m}^{-1}$$

Then

$$T = 16 \left(\frac{0.2}{0.8} \right) \left(1 - \frac{0.2}{0.8} \right) \\ \times \exp \left[-2(1.027 \times 10^9)(15 \times 10^{-10}) \right]$$

or

$$T = 0.138$$

(b) For $m = (1.08)m_o$

$$k_2 =$$

$$\left\{ \frac{2(1.08)(9.11 \times 10^{-31})(0.8 - 0.2)(1.6 \times 10^{-19})}{(1.054 \times 10^{-34})^2} \right\}^{1/2}$$

or

$$k_2 = 4.124 \times 10^9 \text{ m}^{-1}$$

Then

$$T = 16 \left(\frac{0.2}{0.8} \right) \left(1 - \frac{0.2}{0.8} \right) \\ \times \exp \left[-2(4.124 \times 10^9)(15 \times 10^{-10}) \right]$$

or

$$T = 1.27 \times 10^{-5}$$

2.37

$$T \cong 16 \left(\frac{E}{V_o} \right) \left(1 - \frac{E}{V_o} \right) \exp(-2k_2 a)$$

where $k_2 = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$

$$= \frac{\sqrt{2(1.67 \times 10^{-27})(12 - 1) \times 10^6 \times (1.6 \times 10^{-19})}}{1.054 \times 10^{-34}}$$

$$= 7.274 \times 10^{14} \text{ m}^{-1}$$

(a)

$$T \cong 16 \left(\frac{1}{12} \right) \left(1 - \frac{1}{12} \right) \exp \left[-2(7.274 \times 10^{14})(10^{-14}) \right]$$

$$= 1.222 \exp[-14.548]$$

$$= 5.875 \times 10^{-7}$$

(b)

$$T = (10)(5.875 \times 10^{-7}) \\ = 1.222 \exp \left[-2(7.274 \times 10^{14})a \right]$$

$$2(7.274 \times 10^{14})a = \ln \left(\frac{1.222}{5.875 \times 10^{-6}} \right)$$

or $a = 0.842 \times 10^{-14} \text{ m}$

2.38

Region I ($x < 0$), $V = 0$;

Region II ($0 < x < a$), $V = V_o$

Region III ($x > a$), $V = 0$

(a) Region I:

$$\psi_1(x) = A_1 \exp(jk_1 x) + B_1 \exp(-jk_1 x) \\ \text{(incident) \quad (reflected)}$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

Region II:

$$\psi_2(x) = A_2 \exp(k_2 x) + B_2 \exp(-k_2 x)$$

where

$$k_2 = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$$

Region III:

$$\psi_3(x) = A_3 \exp(jk_1 x) + B_3 \exp(-jk_1 x)$$

(b)

In Region III, the B_3 term represents a reflected wave. However, once a particle is transmitted into Region III, there will not be a reflected wave so that $B_3 = 0$.

(c) Boundary conditions:

$$\text{At } x=0: \psi_1 = \psi_2 \Rightarrow$$

$$A_1 + B_1 = A_2 + B_2$$

$$\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx} \Rightarrow$$

$$jk_1 A_1 - jk_1 B_1 = k_2 A_2 - k_2 B_2$$

$$\text{At } x=a: \psi_2 = \psi_3 \Rightarrow$$

$$A_2 \exp(k_2 a) + B_2 \exp(-k_2 a) = A_3 \exp(jk_1 a)$$

$$\frac{d\psi_2}{dx} = \frac{d\psi_3}{dx} \Rightarrow$$

$$k_2 A_2 \exp(k_2 a) - k_2 B_2 \exp(-k_2 a) = jk_1 A_3 \exp(jk_1 a)$$

The transmission coefficient is defined as

$$T = \frac{A_3 A_3^*}{A_1 A_1^*}$$

so from the boundary conditions, we want to solve for A_3 in terms of A_1 . Solving

for A_1 in terms of A_3 , we find

$$A_1 = \frac{+jA_3}{4k_1 k_2} \left\{ (k_2^2 - k_1^2) [\exp(k_2 a) - \exp(-k_2 a)] - 2jk_1 k_2 [\exp(k_2 a) + \exp(-k_2 a)] \right\} \times \exp(jk_1 a)$$

We then find

$$A_1 A_1^* = \frac{A_3 A_3^*}{(4k_1 k_2)^2} \left\{ (k_2^2 - k_1^2) [\exp(k_2 a) - \exp(-k_2 a)]^2 + 4k_1^2 k_2^2 [\exp(k_2 a) + \exp(-k_2 a)]^2 \right\}$$

We have

$$k_2 = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$$

If we assume that $V_o \gg E$, then $k_2 a$ will be large so that

$$\exp(k_2 a) \gg \exp(-k_2 a)$$

We can then write

$$A_1 A_1^* = \frac{A_3 A_3^*}{(4k_1 k_2)^2} \left\{ (k_2^2 - k_1^2) [\exp(k_2 a)]^2 + 4k_1^2 k_2^2 [\exp(k_2 a)]^2 \right\}$$

which becomes

$$A_1 A_1^* = \frac{A_3 A_3^*}{(4k_1 k_2)^2} (k_2^2 + k_1^2) \exp(2k_2 a)$$

Substituting the expressions for k_1 and k_2 , we find

$$k_1^2 + k_2^2 = \frac{2mV_o}{\hbar^2}$$

and

$$\begin{aligned} k_1^2 k_2^2 &= \left[\frac{2m(V_o - E)}{\hbar^2} \right] \left[\frac{2mE}{\hbar^2} \right] \\ &= \left(\frac{2m}{\hbar^2} \right)^2 (V_o - E)(E) \\ &= \left(\frac{2m}{\hbar^2} \right)^2 (V_o) \left(1 - \frac{E}{V_o} \right) (E) \end{aligned}$$

Then

$$\begin{aligned} A_1 A_1^* &= \frac{A_3 A_3^* \left(\frac{2mV_o}{\hbar^2} \right)^2 \exp(2k_2 a)}{16 \left[\left(\frac{2m}{\hbar^2} \right)^2 V_o \left(1 - \frac{E}{V_o} \right) (E) \right]} \\ &= \frac{A_3 A_3^*}{16 \left(\frac{E}{V_o} \right) \left(1 - \frac{E}{V_o} \right) \exp(-2k_2 a)} \end{aligned}$$

Finally,

$$T = \frac{A_3 A_3^*}{A_1 A_1^*} = 16 \left(\frac{E}{V_o} \right) \left(1 - \frac{E}{V_o} \right) \exp(-2k_2 a)$$

2.39

Region I: $V = 0$

$$\frac{\partial^2 \psi_1(x)}{\partial x^2} + \frac{2mE}{\hbar^2} \psi_1(x) = 0 \Rightarrow$$

$$\psi_1(x) = A_1 \exp(jk_1 x) + B_1 \exp(-jk_1 x)$$

incident reflected

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

Region II: $V = V_1$

$$\frac{\partial^2 \psi_2(x)}{\partial x^2} + \frac{2m(E - V_1)}{\hbar^2} \psi_2(x) = 0 \Rightarrow$$

$$\psi_2(x) = A_2 \exp(jk_2 x) + B_2 \exp(-jk_2 x)$$

transmitted reflected

where

$$k_2 = \sqrt{\frac{2m(E - V_1)}{\hbar^2}}$$

Region III: $V = V_2$

$$\frac{\partial^2 \psi_3(x)}{\partial x^2} + \frac{2m(E - V_2)}{\hbar^2} \psi_3(x) = 0 \Rightarrow$$

$$\psi_3(x) = A_3 \exp(jk_3 x)$$

transmitted

where

$$k_3 = \sqrt{\frac{2m(E - V_2)}{\hbar^2}}$$

There is no reflected wave in Region III.
The transmission coefficient is defined as:

$$T = \frac{v_3}{v_1} \cdot \frac{A_3 A_3^*}{A_1 A_1^*} = \frac{k_3}{k_1} \cdot \frac{A_3 A_3^*}{A_1 A_1^*}$$

From the boundary conditions, solve for A_3 in terms of A_1 . The boundary conditions are:

At $x = 0$: $\psi_1 = \psi_2 \Rightarrow$

$$A_1 + B_1 = A_2 + B_2$$

$$\frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x} \Rightarrow$$

$$k_1 A_1 - k_1 B_1 = k_2 A_2 - k_2 B_2$$

At $x = a$: $\psi_2 = \psi_3 \Rightarrow$

$$A_2 \exp(jk_2 a) + B_2 \exp(-jk_2 a) = A_3 \exp(jk_3 a)$$

$$\begin{aligned} \frac{\partial \psi_2}{\partial x} &= \frac{\partial \psi_3}{\partial x} \Rightarrow \\ k_2 A_2 \exp(jk_2 a) - k_2 B_2 \exp(-jk_2 a) &= k_3 A_3 \exp(jk_3 a) \end{aligned}$$

But $k_2 a = 2n\pi \Rightarrow$

$$\exp(jk_2 a) = \exp(-jk_2 a) = 1$$

Then, eliminating B_1 , A_2 , B_2 from the boundary condition equations, we find

$$T = \frac{k_3}{k_1} \cdot \frac{4k_1^2}{(k_1 + k_3)^2} = \frac{4k_1 k_3}{(k_1 + k_3)^2}$$

2.40

(a) Region I: Since $V_0 > E$, we can write

$$\frac{\partial^2 \psi_1(x)}{\partial x^2} - \frac{2m(V_0 - E)}{\hbar^2} \psi_1(x) = 0$$

Region II: $V = 0$, so

$$\frac{\partial^2 \psi_2(x)}{\partial x^2} + \frac{2mE}{\hbar^2} \psi_2(x) = 0$$

Region III: $V \rightarrow \infty \Rightarrow \psi_3 = 0$

The general solutions can be written, keeping in mind that ψ_1 must remain finite for $x < 0$, as

$$\psi_1(x) = B_1 \exp(k_1 x)$$

$$\psi_2(x) = A_2 \sin(k_2 x) + B_2 \cos(k_2 x)$$

$$\psi_3(x) = 0$$

where

$$k_1 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \quad \text{and} \quad k_2 = \sqrt{\frac{2mE}{\hbar^2}}$$

(b) Boundary conditions

At $x = 0$: $\psi_1 = \psi_2 \Rightarrow B_1 = B_2$

$$\frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x} \Rightarrow k_1 B_1 = k_2 A_2$$

At $x = a$: $\psi_2 = \psi_3 \Rightarrow$

$$A_2 \sin(k_2 a) + B_2 \cos(k_2 a) = 0$$

or

$$B_2 = -A_2 \tan(k_2 a)$$

(c)

$$k_1 B_1 = k_2 A_2 \Rightarrow A_2 = \left(\frac{k_1}{k_2} \right) B_1$$

and since $B_1 = B_2$, then

$$A_2 = \left(\frac{k_1}{k_2} \right) B_2$$

From $B_2 = -A_2 \tan(k_2 a)$, we can write

$$B_2 = -\left(\frac{k_1}{k_2}\right) B_2 \tan(k_2 a)$$

or

$$1 = -\left(\frac{k_1}{k_2}\right) \tan(k_2 a)$$

This equation can be written as

$$1 = -\sqrt{\frac{V_o - E}{E}} \cdot \tan\left[\sqrt{\frac{2mE}{\hbar^2}} \cdot a\right]$$

or

$$\sqrt{\frac{E}{V_o - E}} = -\tan\left[\sqrt{\frac{2mE}{\hbar^2}} \cdot a\right]$$

This last equation is valid only for specific values of the total energy E . The energy levels are quantized.

2.41

$$\begin{aligned} E_n &= \frac{-m_o e^4}{(4\pi\epsilon_o)^2 2\hbar^2 n^2} \text{ (J)} \\ &= \frac{-m_o e^3}{(4\pi\epsilon_o)^2 2\hbar^2 n^2} \text{ (eV)} \\ &= \frac{-(9.11 \times 10^{-31})(1.6 \times 10^{-19})^3}{[4\pi(8.85 \times 10^{-12})]^2 2(1.054 \times 10^{-34})^2 n^2} \end{aligned}$$

or

$$\begin{aligned} E_n &= \frac{-13.58}{n^2} \text{ (eV)} \\ n=1 &\Rightarrow E_1 = -13.58 \text{ eV} \\ n=2 &\Rightarrow E_2 = -3.395 \text{ eV} \\ n=3 &\Rightarrow E_3 = -1.51 \text{ eV} \\ n=4 &\Rightarrow E_4 = -0.849 \text{ eV} \end{aligned}$$

2.42

We have

$$\psi_{100} = \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1}{a_o}\right)^{3/2} \exp\left(\frac{-r}{a_o}\right)$$

and

$$\begin{aligned} P &= 4\pi r^2 \psi_{100}^* \psi_{100} \\ &= 4\pi r^2 \cdot \frac{1}{\pi} \cdot \left(\frac{1}{a_o}\right)^3 \exp\left(\frac{-2r}{a_o}\right) \end{aligned}$$

or

$$P = \frac{4}{(a_o)^3} \cdot r^2 \exp\left(\frac{-2r}{a_o}\right)$$

To find the maximum probability

$$\begin{aligned} \frac{dP(r)}{dr} &= 0 \\ &= \frac{4}{(a_o)^3} \left\{ \left(\frac{-2}{a_o}\right) (r^2) \exp\left(\frac{-2r}{a_o}\right) \right. \\ &\quad \left. + 2r \exp\left(\frac{-2r}{a_o}\right) \right\} \end{aligned}$$

which gives

$$0 = \frac{-r}{a_o} + 1 \Rightarrow r = a_o$$

or $r = a_o$ is the radius that gives the greatest probability.

2.43

ψ_{100} is independent of θ and ϕ , so the wave equation in spherical coordinates reduces to

$$\frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{2m_o}{\hbar^2} (E - V(r)) \psi = 0$$

where

$$V(r) = \frac{-e^2}{4\pi\epsilon_o r} = \frac{-\hbar^2}{m_o a_o r}$$

For

$$\psi_{100} = \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1}{a_o}\right)^{3/2} \exp\left(\frac{-r}{a_o}\right)$$

Then

$$\frac{\partial \psi_{100}}{\partial r} = \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1}{a_o}\right)^{3/2} \left(\frac{-1}{a_o}\right) \exp\left(\frac{-r}{a_o}\right)$$

so

$$r^2 \frac{\partial \psi_{100}}{\partial r} = \frac{-1}{\sqrt{\pi}} \cdot \left(\frac{1}{a_o}\right)^{5/2} r^2 \exp\left(\frac{-r}{a_o}\right)$$

We then obtain

$$\begin{aligned} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi_{100}}{\partial r} \right) &= \frac{-1}{\sqrt{\pi}} \cdot \left(\frac{1}{a_o}\right)^{5/2} \\ &\quad \times \left[2r \exp\left(\frac{-r}{a_o}\right) - \left(\frac{r^2}{a_o}\right) \exp\left(\frac{-r}{a_o}\right) \right] \end{aligned}$$

Substituting into the wave equation, we have

$$\begin{aligned} & \frac{-1}{r^2 \sqrt{\pi}} \cdot \left(\frac{1}{a_o} \right)^{5/2} \left[2r \exp\left(\frac{-r}{a_o} \right) - \frac{r^2}{a_o} \exp\left(\frac{-r}{a_o} \right) \right] \\ & + \frac{2m_o}{\hbar^2} \left[E + \frac{\hbar^2}{m_o a_o r} \right] \\ & \times \left(\frac{1}{\sqrt{\pi}} \right) \cdot \left(\frac{1}{a_o} \right)^{3/2} \exp\left(\frac{-r}{a_o} \right) = 0 \end{aligned}$$

where

$$E = E_1 = \frac{-m_o e^4}{(4\pi\epsilon_o)^2 2\hbar^2} = \frac{-\hbar^2}{2m_o a_o^2}$$

Then the above equation becomes

$$\begin{aligned} & \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1}{a_o} \right)^{3/2} \left[\exp\left(\frac{-r}{a_o} \right) \right] \left\{ \frac{-1}{r^2 a_o} \left[2r - \frac{r^2}{a_o} \right] \right. \\ & \left. + \frac{2m_o}{\hbar^2} \left(\frac{-\hbar^2}{2m_o a_o} + \frac{\hbar^2}{m_o a_o r} \right) \right\} = 0 \end{aligned}$$

or

$$\begin{aligned} & \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1}{a_o} \right)^{3/2} \left[\exp\left(\frac{-r}{a_o} \right) \right] \\ & \times \left\{ \frac{-2}{a_o r} + \frac{1}{a_o^2} + \left(\frac{-1}{a_o^2} + \frac{2}{a_o r} \right) \right\} = 0 \end{aligned}$$

which gives $0 = 0$ and shows that ψ_{100} is indeed a solution to the wave equation.

2.44

All elements are from the Group I column of the periodic table. All have one valence electron in the outer shell.