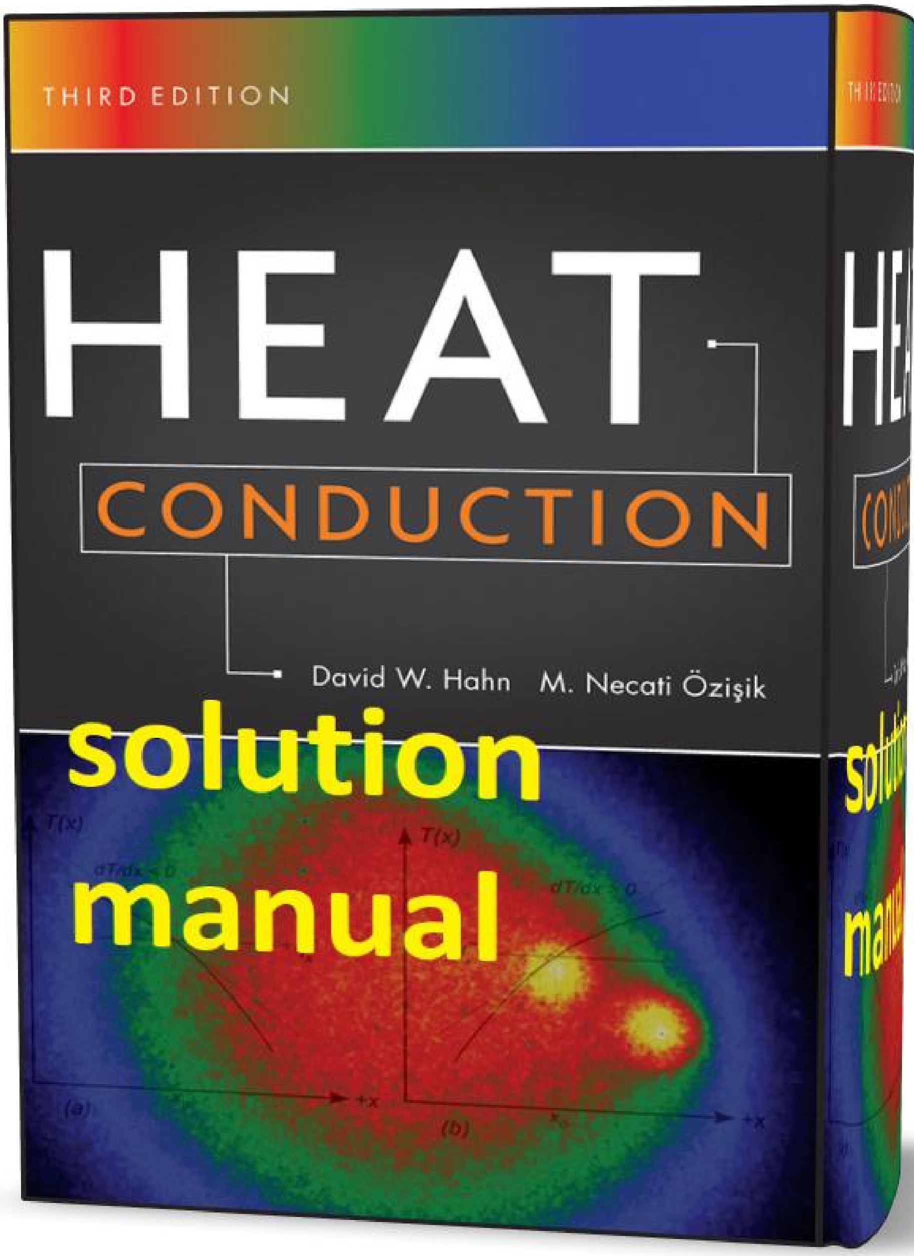
THIRD EDITION
HEAT
CONDUCTION
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solution
manual

1-1 Derive the heat conduction equation (1-43) in cylindrical coordinates using the differential control approach beginning with the general statement of conservation of energy. Show all steps and list all assumptions. Consider Fig. 1-7.

Assume quiescent medium with no mass flow in or out of the control volume.

Assume no work by control volume.

$$\delta \dot{Q} + \delta \dot{E}_{gen} = \frac{dE_{cv}}{dt} \quad \Big\} \text{ cons. of energy}$$

per figure 1-7:

$$dV = r\,d\phi \cdot dr \cdot dz$$

$$dm = \rho\,dV = \rho\,r\,d\phi\,dr\,dz$$

$$q_r = -k A_r \frac{\partial T}{\partial r} \quad \text{with } A_r = r\,d\phi\,dz$$

$$q_z = -k A_z \frac{\partial T}{\partial z} \quad \text{with } A_z = r\,d\phi\,dr$$

$$q_\phi = -k A_\phi \left(\frac{1}{r}\right) \frac{\partial T}{\partial \phi} \quad \text{with } A_\phi = dr\,dz$$

↳ Scale factor.

$$q_{r+dr} = q_r + \frac{\partial}{\partial r}(q_r)\,dr$$

$$q_{z+dz} = q_z + \frac{\partial}{\partial z}(q_z)\,dz$$

$$q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial \phi}(q_\phi)\,d\phi$$

1-1

$$\delta \dot{Q} = q_r - q_{r+dr} + q_z - q_{z+dz} + q_\phi - q_{\phi+d\phi}$$

using the above:

$$\rightarrow \delta \dot{Q} = \frac{\partial}{\partial r}\left(k \cdot r\,d\phi\,dz \frac{\partial T}{\partial r}\right) dr + \frac{\partial}{\partial z}\left(k \cdot r\,d\phi\,dr \frac{\partial T}{\partial z}\right) dz + \frac{\partial}{\partial \phi}\left(k \cdot \frac{dr\,dz}{r} \frac{\partial T}{\partial \phi}\right) d\phi$$

$\delta \dot{E}_{gen} = g \cdot dV$, with g equal to rate of internal energy per unit volume (W/m^3)

$$\rightarrow \delta \dot{E}_{gen} = g \cdot r\,d\phi\,dr\,dz$$

$$\frac{dE_{c.v.}}{dt} = \frac{d}{dt}(dm \cdot u) = \rho\,dV \cdot \frac{du}{dt}$$

with u equal to internal energy per unit mass (J/kg)

$$\rightarrow u = cT + \text{constant}$$

$$\rightarrow \frac{dE}{dt} = \rho\,r\,d\phi\,dr\,dz\,c \frac{\partial T}{\partial t}$$

with C equal to the specific heat ($J/kg\,K$)

Substituting the above three terms into the energy equation:

$$\frac{\partial}{\partial r}\left(k \cdot r\,d\phi\,dz\,\frac{\partial T}{\partial r}\right)dr + \frac{\partial}{\partial z}\left(k \cdot r\,d\phi\,dr\,\frac{\partial T}{\partial z}\right)dz$$

$$+ \frac{\partial}{\partial \phi}\left(k \cdot \frac{dr\,dz}{r}\,\frac{\partial T}{\partial \phi}\right)d\phi + g \cdot r\,dr \cdot d\phi \cdot dz$$

$$= \rho c\left(r\,dr\,d\phi\,dz\right)\frac{\partial T}{\partial t}$$

→ Assume k = constant, and divide all terms by $k \cdot r\,dr\,d\phi\,dz$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

1-2 Derive the heat conduction equation (1-46) in spherical coordinates using the differential control approach beginning with the general statement of conservation of energy. Show all steps and list all assumptions. Consider Fig. 1-8.

Assume quiescent medium with no mass flow in or out of the control volume.

Assume no work by control volume.

Cons. of energy: $$\delta \dot{Q} + \delta \dot{E}_{gen} = \frac{dE_{c.v.}}{dt}$$

per figure 1-8: $$dV = (r\,d\theta)(r\sin\theta\,d\phi)(dr)$$

$$dV = r^2 \sin\theta\, dr\, d\phi\, d\theta$$

$$dm = \rho\, dV$$

$$q_r = -k A_r \frac{\partial T}{\partial r} \quad \text{with } A_r = r^2 \sin\theta\, d\theta\, d\phi$$

$$q_\theta = -\frac{k}{r} A_\theta \frac{\partial T}{\partial \theta} \quad \text{with } A_\theta = r\sin\theta\, d\phi\, dr$$

$$q_\phi = \frac{-k}{r\sin\theta} A_\phi \frac{\partial T}{\partial \phi} \quad \text{with } A_\phi = r\, d\theta\, dr$$

→ noting scale factors of $\left(\frac{1}{r}\right)$ & $\left(\frac{1}{r\sin\theta}\right)$

$$q_{r+dr} = q_r + \frac{\partial}{\partial r}(q_r)\,dr$$

$$q_{\theta+d\theta} = q_\theta + \frac{\partial}{\partial \theta}(q_\theta)\,d\theta$$

$$q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial \phi}(q_\phi)\,d\phi$$

1-2

$$\delta\dot{Q} = q_r - q_{r+dr} + q_\theta - q_{\theta+d\theta} + q_\phi - q_{\phi+d\phi}$$

using the above expressions:

$$\rightarrow \delta\dot{Q} = \frac{\partial}{\partial r}\left(k \cdot r^2 \sin\theta \, d\theta \, d\phi \frac{\partial T}{\partial r}\right) dr + \frac{\partial}{\partial\theta}\left(\frac{k}{r} \cdot r\sin\theta \, d\phi \, dr \frac{\partial T}{\partial\theta}\right) d\theta + \frac{\partial}{\partial\phi}\left(\frac{k}{r\sin\theta} \cdot r \, d\theta \, dr \frac{\partial T}{\partial\phi}\right) d\phi$$

$\delta\dot{E}_{gen} = g \cdot dV$, with g equal to the rate of internal energy per unit volume (W/m^3).

$$\rightarrow \delta\dot{E}_{gen} = g \cdot r^2 \sin\theta \, dr \, d\phi \, d\theta$$

$$\frac{dE_{c.v.}}{dt} = \frac{d}{dt}(dm \cdot u) = \rho \, dV \cdot \frac{du}{dt}$$

with u equal to the internal energy per unit mass (J/kg)

$$\rightarrow u = cT + \text{constant}$$

$$\rightarrow \frac{dE}{dt} = \rho r^2 \sin\theta \, dr \, d\phi \, d\theta \cdot c \frac{\partial T}{\partial t}$$

with c equal to the specific heat $(J/kg\,K)$

1-2

Substituting the above three terms into the energy equation:

$$\frac{\partial}{\partial r}\left(k r^2 \sin\theta \, d\theta \, d\phi \frac{\partial T}{\partial r}\right) dr$$
$$+ \frac{\partial}{\partial \theta}\left(\frac{k}{r} \cdot r \sin\theta \, d\phi \, dr \frac{\partial T}{\partial \theta}\right) d\theta$$
$$+ \frac{\partial}{\partial \phi}\left(\frac{k}{r\sin\theta} \cdot r \, d\theta \, dr \frac{\partial T}{\partial \phi}\right) d\phi$$

$$+ g \cdot r^2 \sin\theta \, dr \, d\phi \, d\theta = \rho c r^2 \cdot \sin\theta \, dr \, d\phi \, d\theta \frac{dT}{dt}$$

Assume k = constant, and divide by $k \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta} \underbrace{\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial \phi}\right)}_{= \frac{\partial^2 T}{\partial \phi^2}} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

1-3 Show that the following two forms of the differential operator in the cylindrical coordinate system are equivalent:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}$$

LHS) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) =$

$$\frac{1}{r}\left\{\frac{\partial r}{\partial r}\cdot\frac{\partial T}{\partial r} + r\cdot\frac{\partial^2 T}{\partial r^2}\right\}$$

$$= \frac{1}{r}\left\{1\cdot\frac{\partial T}{\partial r} + r\cdot\frac{\partial^2 T}{\partial r^2}\right\}$$

$$= \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} = RHS$$

1-4 Show that the following three different forms of the differential operator in the spherical coordinate system are equivalent:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=\frac{1}{r}\frac{d^2}{dr^2}(rT)=\frac{d^2T}{dr^2}+\frac{2}{r}\frac{dT}{dr}$$

LHS) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$

$$=\frac{1}{r^2}\left\{\frac{\partial \cdot r^2}{\partial r}\cdot\frac{\partial T}{\partial r}+r^2\cdot\frac{\partial^2 T}{\partial r^2}\right\}$$

$$=\frac{1}{r^2}\left\{2r\cdot\frac{\partial T}{\partial r}+r^2\frac{\partial^2 T}{\partial r^2}\right\}=\frac{2}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}$$

$$=RHS$$

Middle term) $\frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}=\frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{\partial r}{\partial r}\cdot T+r\frac{\partial T}{\partial r}\right\}$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left\{1\cdot T+r\frac{\partial T}{\partial r}\right\}$$

$$=\frac{1}{r}\left\{\frac{\partial T}{\partial r}+1\cdot\frac{\partial T}{\partial r}+r\cdot\frac{\partial^2 T}{\partial r^2}\right\}$$

$$=\frac{1}{r}\left\{2\cdot\frac{\partial T}{\partial r}+r\frac{\partial^2 T}{\partial r^2}\right\}$$

$$=\frac{2}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}=RHS$$

1-5 Set up the mathematical formulation of the following heat conduction problems. Formulation includes the simplified differential heat equation along with boundary and initial conditions. Do not solve the problems.

1. A slab in $0 \le x \le L$ is initially at a temperature $F(x)$. For times $t > 0$, the boundary at $x = 0$ is kept insulated, and the boundary at $x = L$ dissipates heat by convection into a medium at zero temperature.
2. A semi-infinite region $0 \le x \le \infty$ is initially at a temperature $F(x)$. For times $t > 0$, heat is generated in the medium at a constant, uniform rate of g_0 (W/m^3), while the boundary at $x = 0$ is kept at zero temperature.
3. A hollow cylinder $a \le r \le b$ is initially at a temperature $F(r)$. For times $t > 0$, heat is generated within the medium at a rate of $g(r)$, (W/m^3), while both the inner boundary at $r = a$ and outer boundary $r = b$ dissipate heat by convection into mediums at fluid temperature T_∞.
4. A solid sphere $0 \le r \le b$ is initially at temperature $F(r)$. For times $t > 0$, heat is generated in the medium at a rate of $g(r)$, (W/m^3), while the boundary at $r = b$ is kept at a uniform temperature T_0.

1) $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$ $\quad 0 < x < L \,, \; t > 0$

BC1) $\left.\dfrac{\partial T}{\partial x}\right|_{x=0} = 0$

BC2) $-k \left.\dfrac{\partial T}{\partial x}\right|_{L} = h \left. T \right|_{x=L}$

IC) $T(t=0) = F(x)$

1-5

2) $$\frac{\partial^2 T}{\partial x^2} + \frac{g_0}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad 0 < x < \infty,\ t > 0$$

BC1) $T(x=0) = 0$

IC) $T(t=0) = F(x)$

<u>note:</u> The I.C. is not recovered as $x \to \infty$ due to generation.

3) $$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{g(r)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad a < r < b,\ t > 0$$

BC1) $-k\left.\frac{\partial T}{\partial r}\right|_{r=a} = -h\left[\left.T\right|_{r=a} - T_\infty\right]$

BC2) $-k\left.\frac{\partial T}{\partial r}\right|_{r=b} = +h\left[\left.T\right|_{r=b} - T_\infty\right]$

IC) $T(t=0) = F(r)$

4) $$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{g(r)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad 0 \le r < b,\ t > 0$$

BC1) $T(r \to 0) \Rightarrow$ finite

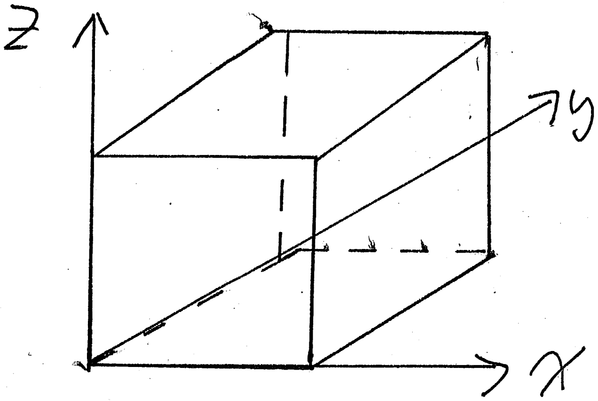
<u>or</u> $\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$ per symmetry

BC2) $T(r=b) = T_0$

IC) $T(t=0) = F(r)$

1-6 A solid cube of dimension L is originally at a uniform temperature T_0. The cube is then dropped into a large bath where the cube rapidly settles flat on the bottom. The fluid in the bath provides convection heat transfer with coefficient h (W/m^2 K) from the fluid at constant temperature T_∞. Formulate the heat conduction problem. Formulation includes the simplified differential heat equation along with appropriate boundary and initial conditions. Include a sketch with your coordinate axis position. Do not solve the problem.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$



$$0 < x < L$$
$$0 < y < L$$
$$0 < z < L$$
$$t > 0$$

BC1) $-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = -h\left[T|_{x=0} - T_\infty\right]$

BC2) $-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = +h\left[T|_{x=L} - T_\infty\right]$

BC3) $-k\left.\frac{\partial T}{\partial y}\right|_{y=0} = -h\left[T|_{y=0} - T_\infty\right]$

BC4) $-k\left.\frac{\partial T}{\partial y}\right|_{y=L} = +h\left[T|_{y=L} - T_\infty\right]$

BC5) $-k\left.\frac{\partial T}{\partial z}\right|_{z=L} = +h\left[T|_{z=L} - T_\infty\right]$

BC6) $T(z=0) = T_\infty$ __or__ $\left.\frac{\partial T}{\partial z}\right|_{z=0} = 0$

→ Convection B.C. not ressonable on the bottom.

IC) $T(t=0) = T_0$

 1/1

1-7 For an anisotropic solid, the three components of the heat conduction vector q_x, q_y and q_z are given by equations (1-80). Write the similar expressions in the cylindrical coordinates for q_r, q_ϕ, q_z and in the spherical coordinates for q_r, q_ϕ, q_θ.

Cylinder

$$q''_r = -\left(k_{11}\frac{\partial T}{\partial r} + k_{12}\frac{1}{r}\frac{\partial T}{\partial \phi} + k_{13}\frac{\partial T}{\partial z}\right)$$

$$q''_\phi = -\left(k_{21}\frac{\partial T}{\partial r} + k_{22}\frac{1}{r}\frac{\partial T}{\partial \phi} + k_{23}\frac{\partial T}{\partial z}\right)$$

$$q''_z = -\left(k_{31}\frac{\partial T}{\partial r} + k_{32}\frac{1}{r}\frac{\partial T}{\partial \phi} + k_{33}\frac{\partial T}{\partial z}\right)$$

Sphere

$$q''_r = -\left(k_{11}\frac{\partial T}{\partial r} + k_{12}\frac{1}{r\cdot\sin\theta}\frac{\partial T}{\partial \phi} + k_{13}\frac{1}{r}\frac{\partial T}{\partial \theta}\right)$$

$$q''_\phi = -\left(k_{21}\frac{\partial T}{\partial r} + k_{22}\frac{1}{r\cdot\sin\theta}\frac{\partial T}{\partial \phi} + k_{23}\frac{1}{r}\frac{\partial T}{\partial \theta}\right)$$

$$q''_\theta = -\left(k_{31}\frac{\partial T}{\partial r} + k_{32}\frac{1}{r\cdot\sin\theta}\frac{\partial T}{\partial \phi} + k_{33}\frac{1}{r}\frac{\partial T}{\partial \theta}\right)$$

1-8 An infinitely long, solid cylinder (D = diameter) has the ability for uniform internal energy generation given by the rate g_o (W/m^3) by passing a current through the cylinder. Initially (t=0), the cylinder is at a uniform temperature T_o. The internal energy generation is then turned on (i.e. current passed) and maintained at a constant rate g_o, and at the same moment the cylinder is exposed to convection heat transfer with coefficient h (W/m^2 K) from a fluid at constant temperature T_∞, noting that $T_\infty > T_o$. The cylinder has uniform and constant thermal conductivity k (W/m K). The Biot number $hD/k \ll 1$. Solve for time t at which point the surface heat flux is exactly zero. Present your answer in variable form.

For $\frac{hD}{k} \ll 1$, use lumped analysis: $T = T(t)$.

$$\cancelto{0}{Q_{in}} + E_{gen} - Q_{out} = \rho V c \frac{\partial T}{\partial t}$$

$$\left(\frac{\pi}{4}D^2 L\right) g_o - (\pi D L) h (T - T_\infty) = \rho\left(\frac{\pi}{4}D^2 L\right) c \frac{\partial T}{\partial t}$$

$$\text{IC)}\quad T(t=0) = T_o$$

now cancel (πL) & let $\theta(t) = T - T_\infty$

$$\left.\begin{aligned} \frac{\partial \theta}{\partial t} + \frac{4h}{\rho D c}\cdot\theta &= \frac{g_o}{\rho c} \\ \theta(t=0) &= T_o - T_\infty \end{aligned}\right\} \quad \text{O.D.E. + I.C.}$$

now $\theta(t) = \theta_H + \theta_P$

$$\Rightarrow \theta(t) = C_1 \cdot e^{-\frac{4h}{\rho D c}t} + \frac{g_o D}{4h}$$

per IC) $\quad C_1 = (T_o - T_\infty) - \frac{g_o D}{4h}$

1-8

2/2

$$\theta(t) = \left[T_o - T_\infty - \frac{g_o D}{4h} \right] e^{-\frac{4h}{\rho D c} t} + \frac{g_o D}{4h}$$

then $$T(t) = \left[T_o - T_\infty - \frac{g_o D}{4h} \right] e^{-\frac{4h}{\rho D c} t} + \frac{g_o D}{4h} + T_\infty$$

Surface flux is zero at $t = t_o$ if $T(t_o) = T_\infty$:

$$T_\infty = \left(T_o - T_\infty - \frac{g_o D}{4h} \right) e^{-\frac{4h}{\rho D c} t_o} + \frac{g_o D}{4h} + T_\infty$$

$$e^{-\frac{4h}{\rho D c} t_o} = \left(\frac{-g_o D}{4h} \right) \frac{1}{\left(T_o - T_\infty - \frac{g D}{4h} \right)}$$

Yields: $$t_o = \left(\frac{-\rho D c}{4h} \right) \ln \left[\frac{\left(\frac{-g_o D}{4h} \right)}{\left(T_o - T_\infty - \frac{g_o D}{4h} \right)} \right]$$

where the term in the brackets is positive for $T_o > T_\infty$.